
COMBINATIONS

Now we discuss the lottery problem mentioned in the Introduction of the chapter *Factorials*.

There are 51 numbers — you choose 6 of them. The order in which you choose the numbers does not make any difference, as long as they're the same 6 numbers selected on the Big Spin. So, to figure the probability of getting all 6 numbers right, we have to ask:



How many ways are there to select 6 numbers from a set of 51 numbers?

Let's put our toes in the water by solving a simpler problem first.

❑ COMBINATIONS

EXAMPLE 1: How many ways are there to choose 3 letters from the set of 5 letters {a, b, c, d, e}?

Solution: Remember, the order doesn't count, so the selection "bce," for instance, is the same as "ecb." A little trial and error yields the following list:

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde

Count them; there are **10 ways** to select 3 letters from the 5.

Can we solve the lottery problem (choosing 6 out of 51) using the same method we just used in Example 1? Sure we could, requiring a boxful of paper and months of writing. There's got to be a formula for this!

Our formula will use the notation $\binom{n}{k}$, read “***n* choose *k***.” It's the number of ways we can choose ***k*** objects from a set of ***n*** objects. The formula, using our friend the factorial, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The “***n* choose *k***” Formula

$\binom{n}{k}$ is also called “a ***combination*** of ***n*** things taken ***k*** at a time.”

Another common notation for $\binom{n}{k}$ is $C(n, k)$.

Let's redo Example 1 using our new combination formula.

EXAMPLE 2: How many ways are there to choose a subcommittee of 3 people from a committee of 5 people?

Solution: We need to calculate “***5 choose 3***.”

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \times 2!} = \frac{120}{6 \times 2} = \frac{120}{12} = 10$$

Is this the answer we obtained in Example 1?

We now have the tools and the talent (I stole this phrase from *Ghostbusters*) to solve the lottery problem. However, we need one more calculation example demonstrating the power of reducing a fraction before we grab our calculators.

EXAMPLE 3: Calculate: $\binom{10}{7}$.

Solution: This is a **combination of 10 things taken 7 at a time** (and order doesn't matter).

$$\begin{aligned}
 \binom{10}{7} &= \frac{10!}{7!(10-7)!} \\
 &= \frac{10!}{7!3!} \\
 &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{[7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1] \times [3 \cdot 2 \cdot 1]} \\
 &= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{[\cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}] \times [3 \cdot 2 \cdot 1]} \\
 &= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \\
 &= \mathbf{120}
 \end{aligned}$$

EXAMPLE 4: **The Lottery Question:** How many ways can 6 numbers be chosen from a set of 51 numbers?

Solution: We apply the combination formula again:

$$\begin{aligned}
 \binom{51}{6} &= \frac{51!}{6!(51-6)!} \\
 &= \frac{51!}{6! \times 45!} \\
 &= \frac{51 \times 50 \times 49 \times 48 \times 47 \times 46 \times 45 \times 44 \times 43 \times \cdots \times 3 \times 2 \times 1}{[6 \times 5 \times 4 \times 3 \times 2 \times 1] \times [45 \times 44 \times 43 \times \cdots \times 3 \times 2 \times 1]} \\
 &= \frac{51 \times 50 \times 49 \times 48 \times 47 \times 46 \times \cancel{45 \times 44 \times 43 \times \cdots \times 3 \times 2 \times 1}}{[6 \times 5 \times 4 \times 3 \times 2 \times 1] \times [\cancel{45 \times 44 \times 43 \times \cdots \times 3 \times 2 \times 1}]} \\
 &= \frac{12,966,811,200}{720} = \mathbf{18,009,460}
 \end{aligned}$$

Don't panic if your calculator goes into scientific notation in the numerator. It will all work out in the end.

Since there are over 18 million ways to select 6 out of 51 numbers, you have about **1 chance in 18 million** to win the lottery. And yes, you could win the lottery — guaranteed — by buying 18,009,460 different lottery tickets. What are the fallacies in this reasoning?

Our final example involves the combination formula using a variable.

EXAMPLE 5: **How many ways are there to select 2 objects out of n objects? (We assume $n \geq 2$.)**

Solution: The combination formula gives us

$$\begin{aligned} \binom{n}{2} &= \frac{n!}{2! \times (n-2)!} = \frac{n(n-1)(n-2)(n-3)\cdots(2)(1)}{[2 \times 1] \times [(n-2)(n-3)\cdots(2)(1)]} \\ &= \frac{n(n-1) \cancel{(n-2)(n-3)\cdots(2)(1)}}{2 \times \cancel{(n-2)(n-3)\cdots(2)(1)}} = \frac{n(n-1)}{2} = \frac{n^2 - n}{2} \end{aligned}$$

Homework

1. Calculate: a. $\binom{7}{3}$ b. $\binom{10}{9}$ c. $\binom{12}{6}$ d. $\binom{20}{12}$
2. Calculate: a. $\binom{100}{98}$ b. $\binom{200}{1}$ c. $\binom{132}{0}$ d. $\binom{750}{750}$
3. Calculate: a. $\binom{n}{0}$ b. $\binom{n}{1}$ c. $\binom{n}{2}$ d. $\binom{n}{3}$
4. Calculate: a. $\binom{n}{n}$ b. $\binom{n}{n-1}$ c. $\binom{n}{n-2}$ d. $\binom{n}{n-3}$

5. a. Show that $\binom{1,000}{250} = \binom{1,000}{750}$.
- b. [Hard] Now try to prove that $\binom{n}{k} = \binom{n}{n-k}$.
6. Calculate:
- a. $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$
- b. $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$
- c. $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$
- d. $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$
- e. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$
7. How many ways are there to choose 5 numbers in a lottery containing the numbers 1 through 100?

Review Problems

8. How many ways are there to choose 12 letters from the first 15 letters of the alphabet, where order does not matter?

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9. In a lottery with 75 numbers, how many ways can you choose 3 numbers?
10. How many ways can you select 3 objects from n objects?
11. Calculate: a. $\binom{197}{197}$ b. $\binom{299}{0}$ c. $\binom{1024}{1}$
12. Prove that $\binom{1000}{123} = \binom{1000}{877}$.
13. True/False:
- a. $\binom{n}{0} = \binom{n}{n}$
- b. $\binom{p}{q} = \binom{p}{p-q}$
- c. There are over 20 million ways to choose 6 numbers from a set of 51 numbers.
- d. There are $\frac{n^2-n}{2}$ ways to choose 2 people out of n people.

Solutions

1. a. 35 b. 10 c. 924 d. 125,970
2. a. 4,950 b. 200 c. 1 d. 1
3. a. 1 b. n c. $\frac{n^2-n}{2}$ d. $\frac{n^3-3n^2+2n}{6}$

4. a. 1 b. n c. $\frac{n^2 - n}{2}$ d. $\frac{n^3 - 3n^2 + 2n}{6}$

5. a. $\binom{1000}{250} = \frac{1000!}{250! \times (1000 - 250)!} = \frac{1000!}{250! \times 750!}$. Also,
 $\binom{1000}{750} = \frac{1000!}{750! \times (1000 - 750)!} = \frac{1000!}{750! \times 250!}$. Since these last two expressions are equivalent (no matter their final numerical values), we've proved the assertion.

b. Like I'm going to give this one away!

6. a. 4 b. 8 c. 16 d. 32
 e. I'm curious what you got.

7. 75,287,520

8. 455 9. 67,525 10. $\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n$

11. a. 1 b. 1 c. 1,024

12. Convince yourself.

13. a. T b. T c. F d. T

